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Research Article

Boundary Control for a Kind of Coupled PDE-ODE System

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A coupled system of an ordinary differential equation (ODE) and a heat partial differential equation (PDE) with spatially varying coefficients is discussed. By using the PDE backstepping method, the state-feedback stabilizing controller is explicitly constructed with the assumptions $\lambda(x) \in P[x]_n$ and $\lambda(x) \in C^{\infty}_{[0,l]}$, respectively. The closed-loop system is proved to be exponentially stable by this controller. A simulation example is presented to illustrate the effectiveness of the proposed method.

1. Introduction

Predictor-feedback control design [1, 2] has been an active area of research for PDE or PDE-ODE coupled control systems [3–5] with actuator and sensor delays that have rich physical backgrounds such as coupled electromagnetic, coupled mechanical, and coupled chemical reactions. The input delays to the ODE system $\dot{X}(t) = AX(t) + BU(t-l)$ can be modeled with the first-order hyperbolic PDE (transport PDE) $u_t(x,t) = u_x(x,t)$ and the boundary condition $u(l,t) = U(t)$. Thus, the original ODE system with input delay can be represented as the following ODE-PDE coupled system (1) that is driven by the input U from the boundary of the PDE:

$$\begin{aligned} \dot{X}(t) &= AX(t) + Bu(0,t) \\ u_t(x,t) &= u_x(x,t) \\ u(l,t) &= U(t). \end{aligned} \tag{1}$$

Control design of coupled PDE-ODE systems was considered in [6–10]. The controller design based on the backstepping method for the coupled system (1) was designed in [9, 10]. More recently in [11], a heat diffusion PDE-ODE coupled system was considered, and a wave PDE-ODE coupled system was considered in [12]. The control system with interaction for this system coupled between the ODE and the PDE was considered in [13]:

$$\begin{aligned} \dot{X}(t) &= AX(t) + Bu(0,t) \\ u_t(x,t) &= u_{xx}(x,t) + CX(t) \\ u_x(0,t) &= 0 \\ u(l,t) &= U(t). \end{aligned} \tag{2}$$

In this system, the ODE acts back on the PDE by the state $X(t)$ of the ODE; meanwhile, the PDE acts on the ODE, which models the solid-gas interaction of heat diffusion and chemical reaction.

In this paper, we replace the spatially constant coefficient C of the PDE subsystem in (2) by the spatially varying coefficient $\lambda(x)$; that is, $u_t(x,t) = u_{xx}(x,t) + \lambda(x)X(t)$, which implies that the effects from the ODE subsystem to the PDE subsystem are varying with the location x. In fact, control of the coupled systems is an important subject in control theory since this type of system arises frequently in control engineering.

The objective of this paper is to convert a PDE-ODE coupled system into a closed-loop target system that is exponentially stable in the sense of the norm $\|u_x\|^2 + \|u\|^2 + |X|^2$, with a designed stable state-feedback controller by using the backstepping-based predictor design method.

Under the assumptions $\lambda(x) \in P[x]_n$ and $\lambda(x) \in C^{\infty}_{[0,l]}$, respectively, we further obtain the explicit expressions of the kernel function of the backstepping transformation.

This paper is organized as follows. In Section 2, we propose the interaction of PDE-ODE coupled control system. In Section 3, a state-feedback boundary controller is designed for this system by the backstepping-based method. In Section 4, we prove the exponential stability for the designed closed-loop system, and Section 5 is a simulation example. In Section 6, some comments are made on the coupled PDE-ODE systems.

Notation. We define the Hilbert space $\Omega = R^{n\times 1} \times H^1(0,l)$ with norm:

$$\|(X,u)\|_{\Omega} = \left(|X|^2 + \int_0^l u^2(x,t)dx + \int_0^l u_x^2(x,t)dx\right)^{1/2}. \tag{3}$$

For n-dimentional vector $\lambda(x) = (\lambda_1(x), \lambda_2(x), \ldots, \lambda_n(x))$, we denote $\lambda(x) \in P[x]_n$ (where $P[x]_n$ is polynomial with n degree) directly with $\lambda_i(x) \in P[x]_n$ $(i = 1,2,\ldots,n)$; we also denote $\lambda(x) \in C^n_{[0,l]}$ if $\lambda_i(x) \in C^n_{[0,l]}$ $(i = 1,2,\ldots, n)$ and $\overline{\lambda} = \max_{x\in[0,l]}|\lambda(x)|$ for $\overline{\lambda} = (\max_{x\in[0,l]}|\lambda_1(x)|, \ldots, \max_{x\in[0,l]}|\lambda_n(x)|)$.

2. Problem Formulation

Consider a coupled interaction system of an ODE and a heat PDE with $\lambda(x)$ times $X(t)$:

$$\begin{gathered}\dot{X}(t) = AX(t) + Bu(0,t)\\ u_t(x,t) = u_{xx}(x,t) + \lambda(x)X(t), \quad x \in (0,l)\\ u_x(0,t) = 0\\ u(l,t) = U(t)\\ X(0) = X_0\\ u(x,0) = u_0(x),\end{gathered} \tag{4}$$

where $X(t) \in R^{n\times 1}$ and $u(x,t)$ are the vector state and scalar state of the ODE subsystem, $X(0) = X_0$ and $u(x,0) = u_0(x)$ are the initial values to the PDE subsystem, respectively, $A \in R^{n\times n}$, $B \in R^{n\times 1}$, and the pair (A,B) are assumed to be stabilization, and $\lambda(x) \in C^{\infty}_{[0,l]}$ is a known spatially varying parameter with $[0,l]$ $(l > 0)$ which is the length of the PDE domain.

From formulation (4), it can be seen that the output $u(0,t)$ of the PDE subsystem acts as the input of the ODE subsystem; meanwhile, the output $X(t)$ of the ODE subsystem affects the PDE along the domain $[0,l]$ times by $\lambda(x)$, which implies that the effects of the ODE subsystem on the PDE subsystem are varying with the location. To design the state-feedback controller for system (4), an infinite-dimensional backstepping method is adopted, which provides an invertible integral transformation $(X,u) \mapsto (X,\omega)$ as follows:

$$\omega(x,t) = u(x,t) - \int_0^x k(x,y)u(y,t)\,dy - \Phi(x)X(t). \tag{5}$$

This transformation converts the plant (4) into the following target system:

$$\begin{gathered}\dot{X}(t) = (A+BK)X(t) + B\omega(0,t)\\ \omega_t(x,t) = \omega_{xx}(x,t)\\ \omega_x(0,t) = 0\\ \omega(l,t) = 0\\ X(0) = X_0\\ \omega(x,0) = \omega_0(x),\end{gathered} \tag{6}$$

where $K \in R^{1\times n}$ satisfying $A + BK$ is Hurwitz. It should be pointed that it is nontrivial to obtain the kernel function $k(x,y)$ by the method in the literature [13] as it is related to $\lambda(x)$. But if we impose some constraints on $\lambda(x)$, then the kernel can be obtained explicitly, as shown in the next section. Once transformation (5) is obtained (namely, $k(x,y)$ and $\Phi(x)$ are obtained), then the controller subject to the boundary condition $\omega(l,t) = 0$ in (6) is given in the form

$$u(l,t) = U(t) = \int_0^l k(l,y)u(y,t)\,dy + \Phi(l)X(t). \tag{7}$$

3. The Design of the State-Feedback Controller

In this section, we will design the predictor-feedback controller $U(t)$ for the system (4) using the backstepping method. The key point to this design is to determine the function $k(x,y)$ and $\Phi(x)$ of transformation (5). In the following procedure, we find that the kernel function $k(x,y)$ can be expressed by $\Phi(x)$, while the function $\Phi(x)$ is related to the spatially varying parameter $\lambda(x)$.

3.1. Preliminary. In order to obtain the explicit solution of $k(x,y)$ and $\Phi(x)$, we will firstly solve the following equation of $\Phi(x)$ with assumptions $\lambda(x) \in P[x]_n$ and $\lambda(x) \in C^{\infty}_{[0,l]}$, respectively:

$$\begin{gathered}\Phi''(x) - \Phi(x)A - \int_0^x\int_0^{x-y}\Phi(z)B\,dz\,\lambda(y)\,dy + \lambda(x) = 0\\ \Phi(0) = K\\ \Phi'(0) = 0.\end{gathered} \tag{8}$$

Lemma 1. *let $\Phi(x) \in R^{1\times n}$, $\lambda(x) \in R^{1\times n}$, $A \in R^{n\times n}$, and $B \in R^{n\times 1}$, let $\lambda(x) \in P[x]_n$, and then (8) about $\Phi(x)$ has the unique solution.*

Proof. It follows easily from (8) that

$$\Phi''(x) - \Phi(x)A + \lambda(x) = \int_0^x\int_0^{x-y}\Phi(z)B\,dz\,\lambda(y)\,dy. \tag{9}$$

By taking the derivative on both sides about x and applying variable substitution, we have

$$\Phi'''(x) - \Phi'(x)A + \lambda'(x) = \int_0^x \Phi(x-y)B\lambda(y)\,dy = \int_0^x \Phi(y)B\lambda(x-y)\,dy. \tag{10}$$

Furthermore, take the derivative repeatedly, and then

$$\Phi^{(n+4)}(x) - \Phi^{(n+2)}(x)A = \Phi^{(n)}(x)B\lambda(0) + \cdots + \Phi(x)B\lambda^{(n)}(0). \tag{11}$$

By this equation, we have

$$\begin{aligned}
&\Phi'(x) = \Phi'(x)\\
&\Phi''(x) = \Phi''(x)\\
&\vdots\\
&\Phi^{(n+3)}(x) = \Phi^{(n+3)}(x)\\
&\Phi^{(n+4)}(x) = \Phi^{(n+2)}(x)A + \Phi^{(n)}(x)B\lambda(0) + \cdots + \Phi(x)B\lambda^{(n)}(0),
\end{aligned} \tag{12}$$

and with the initial values of (8), we have following initial values:

$$\begin{aligned}
&\Phi(0) = K\\
&\Phi'(0) = 0\\
&\Phi''(0) = \Phi(0)A - \lambda(0)\\
&\Phi'''(0) = \Phi'(0)A - \lambda'(0)\\
&\vdots\\
&\Phi^{(n+3)}(0) = \Phi^{(n+1)}(0)A + \Phi^{(n-1)}(0)B\lambda(0) + \cdots + \Phi(0)B\lambda^{(n-1)}(0).
\end{aligned} \tag{13}$$

Then the solution to the ODEs (12) with the above initial values can be expressed as follows:

$$\Phi(x) = \Gamma(0)e^{Dx}E \tag{14}$$

with $\Gamma(0) = (K\ 0\ \Phi''(0)\ \cdots\ \Phi^{(n+2)}(0)\ \Phi^{(n+3)}(0))_{1\times[n(n+4)]}$, where the initial values of higher-order derivatives of Φ in zero are denoted by (13) and

$$D = \begin{pmatrix}
0_{n\times n} & \cdots & \cdots & \cdots & \cdots & 0_{n\times n} & B\lambda^{(n)}(0)\\
I_{n\times n} & & & & & & B\lambda^{(n-1)}(0)\\
 & \ddots & & & 0 & & \vdots\\
 & & I_{n\times n} & & & & B\lambda(0)\\
 & & & I_{n\times n} & & & 0_{n\times n}\\
 & 0 & & & I_{n\times n} & & A\\
 & & & & & I_{n\times n} & 0_{n\times n}
\end{pmatrix}_{[n\times(n+4)]\times[n\times(n+4)]},$$
$$E = \begin{pmatrix} I_{n\times n}\\ 0_{n\times n}\\ \vdots\\ 0_{n\times n}\end{pmatrix}_{[n\times(n+4)]\times n}. \tag{15}$$

□

Lemma 2. *Let $\Phi(x) \in R^{1\times n}$, $\lambda(x) \in R^{1\times n}$, $A \in R^{n\times n}$, $B \in R^{n\times 1}$, and $\lambda(x) \in C^{\infty}_{[0,l]}$, and then* (8) *has the following unique solution:*

$$\Phi(x) = \sum_{n=0}^{+\infty} \Delta\Phi^n(x), \tag{16}$$

$\Delta\Phi^0(x) = K - \int_0^x \int_0^\tau \lambda(\xi)d\xi\, d\tau \Delta\Phi^{n+1}(x) = F[\Delta\Phi^n](x)$, $n = 0, 1, 2, \ldots$

$$F[\Delta\Phi^n](x) = \int_0^x \int_0^\tau \left\{\Delta\Phi^n(\xi)A + \left[\int_0^\xi \left(\int_0^{\xi-y} \Delta\Phi^n(z)B\,dz\right)\lambda(y)\,dy\right]\right\} d\xi\, d\tau. \tag{17}$$

Proof. By integrating into (8) over $[0, x]$, we have

$$\Phi'(x) = \int_0^x \left\{ \Phi(\xi) A + \left[\int_0^\xi \left(\int_0^{\xi - y} \Phi(z) B dz \right) \lambda(y) dy \right] \right\} d\xi - \int_0^x \lambda(\xi) d\xi. \tag{18}$$

Then, integrating again on both sides of (18) over $[0, x]$ with initial value $\Phi(0) = K$ of (8), we conclude

$$\Phi(x) = \Delta\Phi^0(x) + F[\Phi](x), \tag{19}$$

where $F[\Phi](x) = \int_0^x \int_0^\tau \{\Phi(\xi)A + [\int_0^\xi (\int_0^{\xi-y} \Phi(z)Bdz)\lambda(y)dy]\} d\xi\, d\tau$ and $\Delta\Phi^0(x) = K - \int_0^x \int_0^\tau \lambda(\xi)d\xi\, d\tau$.

Denoting the following iterative relationship:

$$\Phi^{n+1}(x) = \Delta\Phi^0(x) + F[\Phi^n](x), \quad n = 0, 1, 2, \ldots, \tag{20}$$

it suffices to show that if series $\{\Phi^n(x)\}$ was convergence, then (16) is the unique solution of (19). Considering the difference

$$\begin{aligned}\Delta\Phi^{n+1}(x) &= \Phi^{n+1}(x) - \Phi^n(x) \\ &= F[\Phi^n](x) - F[\Phi^{n-1}](x) = F[\Delta\Phi^n](x),\end{aligned} \tag{21}$$

now, we will estimate $\Delta\Phi^n(x)$ by induction. First, for $\Delta\Phi^0(x)$, we have

$$\left|\Delta\Phi^0(x)\right| \le |K| + \overline{\lambda}\frac{l^2}{2} = M, \tag{22}$$

where $\overline{\lambda} = \max_{x\in[0,l]}|\lambda(x)|$ is denoted as above and l is the length of the PDE domain. Then, suppose that

$$\left|\Delta\Phi^n(x)\right| \le MN^n \frac{x^{2n}}{(2n)!} \tag{23}$$

with $N = |A| + l^2|B|\overline{\lambda}$. Then $|\Psi^{n+1}(x)|$ can be estimated as follows:

$$\begin{aligned}\left|\Delta\Phi^{n+1}(x)\right| &\le M\frac{N^n}{(2n)!}|A| \int_0^x \int_0^\tau \xi^{2n} d\xi\, d\tau \\ &\quad + M|B|\overline{\lambda}\frac{N^n}{(2i)!} \int_0^x \int_0^\tau \int_0^\xi \int_0^{\xi-y} z^{2n} dz\, dy\, d\xi\, d\tau \\ &\le M\frac{N^{n+1}x^{2n+2}}{(2n+2)!}.\end{aligned} \tag{24}$$

Noting $x \in [0, l]$, we have

$$|\Phi(x)| \le \sum_{n=0}^{\infty} \left|\Delta\Phi^n(x)\right| \le \sum_{n=0}^{+\infty} MN^n \frac{l^{2n}}{(2n)!} \le Me^{l\sqrt{N}}. \tag{25}$$

The series on the right-hand side of (25) converges. Hence by Weierstrass's Discriminance, the series defined by (16) converges absolutely and uniformly on $0 \le x \le l$. Then the existence of the solution to (8) is concluded. To show the uniqueness of the solution (16) to (8), we assume that $\overline{\Phi}$ and $\widetilde{\Phi}$ are two different solutions of (8). Substituting these two solutions and after some direct calculation, we have

$$\delta\Phi(x) = \overline{\Phi} - \widetilde{\Phi} = F[\delta\Phi](x). \tag{26}$$

From (25), we know that $|\Phi(x)| \le Me^{l\sqrt{N}}$, which means $|\delta\Phi(x)| \le 2Me^{l\sqrt{N}}$. Next, we will estimate $\delta\Phi$ by induction. After some direct calculation, we have

$$\delta\Phi^{n+1}(x) = F[\delta\Phi^n](x). \tag{27}$$

Suppose that $|\delta\Phi^n(x)| \le 2Me^{l\sqrt{N}}N^n(x^{2n}/(2n)!)$, and then

$$\begin{aligned}\left|\delta\Phi^{n+1}(x)\right| &\le 2Me^{l\sqrt{N}}\frac{N^n}{(2n)!}|A| \int_0^x \int_0^\tau \xi^{2n} d\xi\, d\tau \\ &\quad + 2Me^{l\sqrt{N}}\frac{N^n}{(2n)!}|B|\overline{\lambda} \\ &\quad \times \int_0^x \int_0^\tau \int_0^\xi \int_0^{\xi-y} z^{2n} dz\, dy\, d\xi\, d\tau \\ &\le 2Me^{l\sqrt{N}}\frac{N^{n+1}x^{2n+2}}{(2n+2)!},\end{aligned} \tag{28}$$

which implies the trueness of $|\delta\Phi^n(x)| \le 2Me^{l\sqrt{N}}N^n(l^{2n}/(2n)!)$. Moreover, since $\lim_{n\to+\infty}2Me^{D\sqrt{N}}N^n(l^{2n}/(2n)!) = 0$, $\delta\Phi(x) \equiv 0$ is easily concluded. Then $\overline{\Phi} = \widetilde{\Phi}$, and (16) is the unique solution to (18). □

3.2. Design of the State-Feedback Controller with Backstepping Method. Next, we will obtain the backstepping transformation (5). Let $x = 0$ in (5), and we have

$$\omega(0, t) = u(0, t) - \Phi(0)X(t), \tag{29}$$

and $\Phi(0) = K$ by comparing (4) and (6). The partial derivatives of $\omega(x, t)$ in (5) with respect to x are given by

$$\begin{aligned}\omega_x(x, t) &= u_x(x, t) - k(x, x)u(x, t) \\ &\quad - \int_0^x k_x(x, y)u(y, t)dy - \Phi'(x)X(t),\end{aligned} \tag{30}$$

$$\begin{aligned}\omega_{xx}(x, t) &= u_{xx}(x, t) - k(x, x)u_x(x, t) \\ &\quad - \left(\frac{d}{dx}k(x, x)\right)u(x, t) - k_x(x, x)u(x, t) \\ &\quad - \int_0^x k_{xx}(x, y)u(y, t)dy - \Phi''(x)X(t).\end{aligned} \tag{31}$$

The derivative of $\omega(x,t)$ with respect to t is

$$\begin{aligned}\omega_t(x,t) &= u_{xx}(x,t) + \lambda(x) X(t) - k(x,x) u_x(x,t) \\ &\quad + k(x,0) u_x(0,t) + k_y(x,x) u(x,t) \\ &\quad - k_y(x,0) u(0,t) - \int_0^x k_{yy}(x,y) u(y,t) dy \\ &\quad - \int_0^x k(x,y) \lambda(y) X(t) dy \\ &\quad - \Phi(x)(AX(t) + Bu(0,t)).\end{aligned} \tag{32}$$

By the target system (6), (31), and (32), we have

$$\begin{aligned}\omega_t(x,t) - \omega_{xx}(x,t) &= 2\left(\frac{d}{dx}k(x,x)\right) u(x,t) \\ &\quad + \left(\lambda(x) - \int_0^x k(x,y)\lambda(y) dy \right. \\ &\quad \left. - \Phi(x) A + \Phi''(x)\right) X(t) \\ &\quad + \int_0^x \left(k_{xx}(x,y) - k_{yy}(x,y)\right) \\ &\quad \times u(y,t) dy \\ &\quad + \left(-k_y(x,0) - \Phi(x) B\right) u(0,t) = 0.\end{aligned} \tag{33}$$

This equation should be valid for all u and X, so we have the following four equations:

$$\begin{gathered}\frac{d}{dx}k(x,x) = 0 \\ \lambda(x) - \int_0^x k(x,y)\lambda(y) dy - \Phi(x) A + \Phi''(x) = 0 \\ k_{xx}(x,y) - k_{yy}(x,y) = 0 \\ -\frac{d}{dy}k(x,0) - \Phi(x) B = 0.\end{gathered} \tag{34}$$

Let $x = 0$ in (30), which gives

$$\omega_x(0,t) = -k(0,0) u(0,t) - \Phi'(0) X(t). \tag{35}$$

Substituting this expression into the boundary condition in (6), we have

$$-k(0,0) u(0,t) - \Phi'(0) X(t) = 0. \tag{36}$$

This equation should be valid for all u and X, so we have two conditions that $k(0,0) = 0$ and $\Phi'(0) = 0$. In order to satisfy the conditions of the target system (6), the $k(x,y)$ and $\Phi(x)$ in (5) should satisfy

$$\begin{gathered}k_{xx}(x,y) = k_{yy}(x,y) \\ k_y(x,0) = -\Phi(x) B \\ \frac{d}{dx}k(x,x) = 0 \\ k(0,0) = 0,\end{gathered} \tag{37}$$

$$\begin{gathered}\lambda(x) - \int_0^x k(x,y)\lambda(y) dy - \Phi(x) A + \Phi''(x) = 0 \\ \Phi(0) = K \\ \Phi'(0) = 0.\end{gathered} \tag{38}$$

Note that (37) is a second-order hyperbolic PDE about $k(x,y)$ and the boundary condition is related to $\Phi(x)$, and (38) is a second-order integral-differential equation about $\Phi(x)$ associated with $\lambda(x)$ and $k(x,y)$. Next, we will obtain $k(x,y)$ from (37) and $\Phi(x)$ from (38).

Suppose $k(x,y) = \Theta(x-y) + \Upsilon(x+y)$; it can be easily obtained by (37) that

$$k(x,y) = \int_0^{x-y} \Phi(z) B\, dz. \tag{39}$$

Substituting this expression into (38), we get

$$\begin{gathered}\lambda(x) - \int_0^x \int_0^{x-y} \Phi(z) B dz \lambda(y) dy - \Phi(x) A + \Phi''(x) = 0 \\ \Phi(0) = K \\ \Phi'(0) = 0.\end{gathered} \tag{40}$$

For $\lambda(x) \in P[x]_n$, by Lemma 1, (40) has a unique solution as follows:

$$\Phi(x) = \Gamma(0) e^{Dx} E, \tag{41}$$

and thus the kernel function $k(x,y)$ can be expressed as follows by (39):

$$k(x,y) = \int_0^{x-y} \Gamma(0) e^{Dz} EB\, dz. \tag{42}$$

For $\lambda(x) \in C^{\infty}_{[0,D]}$, by Lemma 2, (40) has a unique series solution as follows:

$$\Phi(x) = \sum_{n=0}^{+\infty} \Delta\Phi^n(x), \tag{43}$$

and thus the kernel function $k(x,y)$ can be expressed as follows by (39):

$$k(x,y) = \int_0^{x-y} \sum_{n=0}^{+\infty} \Delta\Phi^n(x) B\, dz. \tag{44}$$

Next, we will obtain the inverse transformation of (5) by using a process similar to the one we used above in obtaining the kernels $k(x, y)$ and $\Phi(x)$. Actually, the inverse of the transformation $(X, \omega) \mapsto (X, u)$ can be found as follows:

$$u(x,t) = \omega(x,t) + \int_0^x \iota(x,y)\,\omega(y,t)\,dy + \Psi(x)\,X(t). \tag{45}$$

The kernel functions $\iota(x, y)$ and $\Psi(x)$ can be easily obtained by a method similar to that above

$$\begin{aligned} \iota(x,y) &= \int_0^{x-y} \Psi(z)\,B dz, \\ \Psi(x) &= \left(\Xi(0)\,e^{Dx} + \int_0^x Ce^{F(x-\tau)}d\tau\right)G, \end{aligned} \tag{46}$$

where $\Xi(0) = (K \;\; 0_{n\times n})_{n\times(2n)}$, $F = \begin{pmatrix} 0_{n\times n} & A+BK \\ I_{n\times n} & 0_{n\times n} \end{pmatrix}_{(2n)\times(2n)}$, $C = (0_{n\times n} \;\; \lambda(x))_{n\times(2n)}$, and $G = \begin{pmatrix} I_{n\times n} \\ 0_{n\times n} \end{pmatrix}_{(2n)\times n}$.

Evaluating (5) at $x = l$, and by the boundary condition of (4) and (6), a controller is obtained as follows:

$$U(t) = \int_0^l k(l,y)\,u(y,t)\,dy + \Phi(l)\,X(t). \tag{47}$$

Furthermore, the explicit solution to the closed-loop system (6) under the controller (47) can also be obtained if the initial state $(X(0), u(x, 0))$ is known. The solution in (6) is

$$\begin{aligned} X(t) &= X(0)\,e^{(A+BK)t} + \int_0^t e^{(A+BK)(t-\tau)} B\omega(0,\tau)\,d\tau \\ \omega(x,t) &= \frac{2}{l}\sum_{m=1}^{\infty}\int_0^l \omega_0(\xi)\cos\left(\frac{(m+1/2)\pi}{l}\xi\right)d\xi \\ &\quad\times e^{-((m+1/2)^2\pi^2/l^2)t} \\ &\quad\times \cos\left(\frac{(m+1/2)\pi}{l}x\right) \end{aligned} \tag{48}$$

and the initial condition $\omega_0(x)$ is calculated by the initial state $(X(0), u(x, 0))$ through (5).

4. Exponential Stability of the Coupled PDE-ODE System

Now we will prove the exponential stability of the proposed coupled PDE-ODE system.

Theorem 3. *Let $u(x, 0)$ be a square integrable in x and compatible with the control law (47) (i.e., $u(l, 0) = U(0)$), $X(0) \in R^{n\times1}$, and $u(\cdot, 0) \in H^1(0, l)$, and then the closed-loop system consisting of the plant (4) with the control law (47) has a unique solution $(X(t), u(x, t)) \in \Omega([0, \infty))$ and is exponentially stable in the sense of the norm (3).*

Proof. Consider the Lyapunov function

$$V = X^T PX + \frac{a}{2}\|\omega(\cdot,t)\|^2 + \frac{1}{2}\|\omega_x(\cdot,t)\|^2, \tag{49}$$

where the matrix $P = P^T > 0$ is the solution to the Lyapunov equation $P(A + BK) + (A + BK)^T P = -Q$ for some $Q = Q^T > 0$ and $a > 0$ is a parameter chosen later. By the Cauchy inequality and the Hölder inequality in (5) and (45), we have

$$\begin{aligned} \|\omega\|^2 &\le \alpha_1\|u\|^2 + \alpha_2|X|^2 \\ \|u\|^2 &\le \beta_1\|\omega\|^2 + \beta_2|X|^2, \end{aligned} \tag{50}$$

where

$$\begin{aligned} \alpha_1 &= 3\left(1 + l\max_{0\le x,y\le l}\|k\|^2\right), \qquad \alpha_2 = 3|\Phi|^2 \\ \beta_1 &= 3\left(1 + l\max_{0\le x,y\le l}\|\iota\|^2\right), \qquad \beta_2 = 3|\Psi|^2, \end{aligned} \tag{51}$$

$$\begin{aligned} \|\omega_x\|^2 &\le \alpha_3\|u_x\|^2 + \alpha_4\|u\|^2 + \alpha_5|X|^2 \\ \|u_x\|^2 &\le \beta_3\|\omega_x\|^2 + \beta_4\|\omega\|^2 + \beta_5|X|^2, \end{aligned} \tag{52}$$

where

$$\begin{gathered} \alpha_3 = 4, \qquad \alpha_4 = 4\max_{0\le x,y\le l}\|k\|^2 + 4l\max_{0\le x,y\le l}\|k_x\|^2, \\ \alpha_5 = 4\left|\Phi'\right|^2 \\ \beta_3 = 4, \qquad \beta_4 = 4\max_{0\le x,y\le l}\|\iota\|^2 + 4l\max_{0\le x,y\le l}\|\iota_x\|^2 \beta_5 = 4\left|\Psi'\right|^2. \end{gathered} \tag{53}$$

Combining (50) with (52), we obtain

$$\begin{aligned} V &\le \lambda_{\max}(P)\,|X|^2 + \frac{a}{2}\alpha_1\|u\|^2 + \frac{a}{2}\alpha_2|X|^2 \\ &\quad + \frac{1}{2}\alpha_3\|u_x\|^2 + \frac{1}{2}\alpha_4\|u\|^2 + \frac{1}{2}\alpha_5|X|^2 \\ &\le \overline{\delta}\left(\|u_x\|^2 + \|u\|^2 + |X|^2\right), \end{aligned} \tag{54}$$

where $\overline{\delta} = \max\left\{\lambda_{\max}(P) + \frac{a}{2}\alpha_2 + \frac{1}{2}\alpha_5, \frac{a}{2}\alpha_1 + \frac{1}{2}\alpha_4, \frac{1}{2}\alpha_3\right\}$. Similarly

$$\begin{aligned} &\min\left\{\lambda_{\min}(P), \frac{a}{2}, \frac{1}{2}\right\}\left(\|u_x\|^2 + \|u\|^2 + |X|^2\right) \\ &\quad\le \min\left\{\lambda_{\min}(P), \frac{a}{2}, \frac{1}{2}\right\} \\ &\qquad\times\left(\beta_1\|\omega\|^2 + \beta_2|X|^2 + \beta_3\|\omega_x\|^2\right. \\ &\qquad\left. + \beta_4\|\omega\|^2 + \beta_5|X|^2 + |X|^2\right) \\ &\quad\le \max\left\{\beta_2 + \beta_5 + 1, \beta_1 + \beta_4, \beta_3\right\} \\ &\qquad\times\left(X^T PX + \frac{a}{2}\|\omega(\cdot,t)\|^2 + \frac{1}{2}\|\omega_x(\cdot,t)\|^2\right), \end{aligned} \tag{55}$$

and then

$$V \ge \delta\left(\|u_x\|^2 + \|u\|^2 + |X|^2\right), \tag{56}$$

where $\delta = \min\{\lambda_{\min}(P), a/2, 1/2\}/\max\{\beta_2+\beta_5+1, \beta_1+\beta_4, \beta_3\}$. By (54) and (56), we have

$$\delta\left(\left\|u_x\right\|^2+\|u\|^2+|X|^2\right) \le V \le \overline{\delta}\left(\left\|u_x\right\|^2+\|u\|^2+|X|^2\right). \tag{57}$$

Taking a derivative of the Lyapunov function along with the solution to the system (6) by using Poincare's inequality, we have

$$\begin{aligned}\dot{V} &= X^T\left((A+BK)^TP+P(A+BK)\right)X+2X^TPB\omega(0,t)\\ &\quad -a\int_0^l \omega_x^2(x,t)\,dx-\int_0^l \omega_{xx}^2(x,t)\,dx\\ &\le -X^TQX+\frac{2}{b}\left|X^TPB\right|^2+\frac{b}{2}\omega(0,t)^2\\ &\quad -a\int_0^l \omega_x^2(x,t)\,dx-\int_0^l \omega_{xx}^2(x,t)\,dx.\end{aligned} \tag{58}$$

If we choose $b = 4\lambda_{\max}(PBB^TP^T)/\lambda_{\min}(Q)$, then

$$\begin{aligned}\dot{V} &\le -\frac{\lambda_{\min}(Q)}{2}|X|^2+\frac{2\lambda_{\max}\left(PBB^TP^T\right)}{\lambda_{\min}(Q)}\omega(0,t)^2\\ &\quad -a\int_0^l \omega_x^2(x,t)\,dx-\int_0^l \omega_{xx}^2(x,t)\,dx.\end{aligned} \tag{59}$$

Applying Poincare's inequality and Ammon's inequality with $w_x(0,t)=0$ and $w(l,t)=0$, we have

$$\omega^2(0,t) \le 4l\int_0^l w_x^2(x,t)\,dx, \tag{60}$$

so

$$\begin{aligned}\dot{V} &\le -\frac{\lambda_{\min}(Q)}{2}|X|^2+\frac{2\lambda_{\max}\left(PBB^TP^T\right)}{\lambda_{\min}(Q)}4l\int_0^l w_x^2(x,t)\,dx\\ &\quad -a\int_0^l \omega_x^2(x,t)\,dx-\frac{1}{4l^2}\int_0^l \omega_x^2(x,t)\,dx\\ &\le -\frac{\lambda_{\min}(Q)}{2}|X|^2\\ &\quad -\frac{1}{1+4l^2}\left(a+\frac{1}{4l^2}-\frac{8l\lambda_{\max}\left(PBB^TP^T\right)}{\lambda_{\min}(Q)}\right)\\ &\quad \times\left(\int_0^l w_x^2(x,t)\,dx+\int_0^l w^2(x,t)\,dx\right).\end{aligned} \tag{61}$$

By taking $a > 8l\lambda_{\max}(PBB^TP^T)/\lambda_{\min}(Q)-1/4l^2$, we obtain

$$\dot{V} \le -\gamma V, \tag{62}$$

where

$$\begin{aligned}0<\gamma=\min&\left\{\frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)}, \frac{2}{1+4l^2}\right.\\ &\times\left(1+\frac{1}{4al^2}-\frac{8l\lambda_{\max}\left(PBB^TP^T\right)}{a\lambda_{\min}(Q)}\right), \frac{2}{1+4l^2}\\ &\left.\times\left(a+\frac{1}{4l^2}-\frac{8l\lambda_{\max}\left(PBB^TP^T\right)}{\lambda_{\min}(Q)}\right)\right\}.\end{aligned} \tag{63}$$

Hence,

$$\left\|u_x\right\|^2+\|u\|^2+|X|^2 \le \delta\left(\left\|u_x\right\|^2+\|u\|^2+|X|^2\right)e^{-\gamma t} \tag{64}$$

for all $t \ge 0$, where $\delta = \overline{\delta}/\delta$. This completes the proof. □

5. A Simulation Example

In this section, an example is given to verify the effectiveness of theoretical results for the following simple system:

$$\begin{gathered}\dot{X}(t) = X(t)+u(0,t)\\ u_t(x,t) = u_{xx}(x,t)+2xX(t)\\ u_x(0,t) = 0\\ u(l,t) = U(t)\\ u(x,0) = x+1\\ X(0) = 1,\end{gathered} \tag{65}$$

where $X(t) \in R, 0 \le l \le 0.1$, and $t \ge 0$. In order to show the transient performance of the closed-loop system, a numerical simulation is executed in Matlab. By using the explicit forward Euler method with 1-step discretization in space, simulation Figures 1 and 2 show that both the states $X(t)$ and $u(x,t)$ converge to zero, which indicates that the closed-loop system is exponentially stable.

The convergence rate to zero for the closed-loop system is determined by the eigenvalues of the PDE-ODE system (6). These eigenvalues are the union of the eigenvalues of $A+BK$, which are placed at desirable locations by the control vector K and of the eigenvalues of the heat equation with a Neumann boundary condition on one end and a Dirichlet boundary condition on the other end. While exponentially stable, the heat equation PDE need not necessarily have fast decay. Fortunately, the compensated actuator dynamics, that is, the w-dynamics in [6], can be sped up arbitrarily by a modified controller [11, 14].

6. Conclusions

In this paper we have developed an explicit controller for a coupled PDE-ODE system with Dirichlet interconnection $Bu(0,t)$, extending the results in [2, 3]. Many open problems

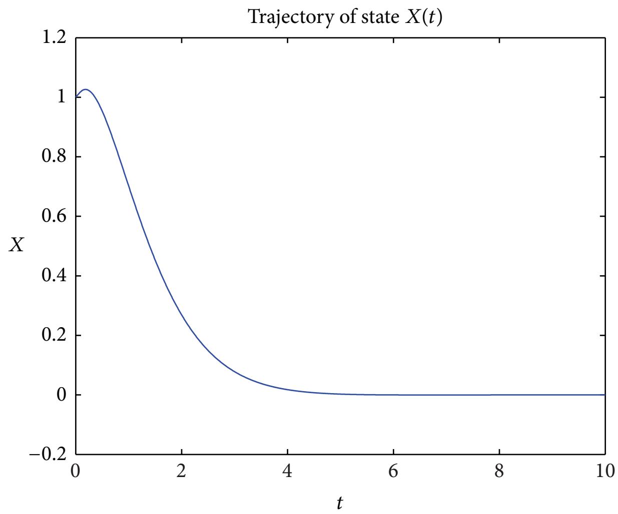

FIGURE 1: Trajectory of the state $X(t)$.

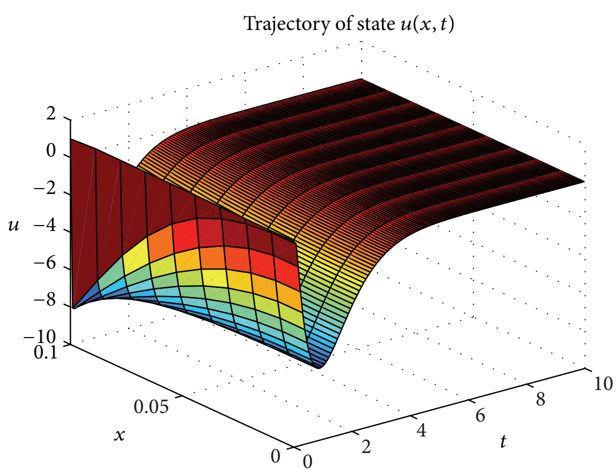

FIGURE 2: Trajectory of the state $u(x,t)$.

in PDE-ODE systems remain. For example, in the system with Neumann interconnection $Bu_x(0,t)$

$$\begin{aligned}\dot{X}(t) &= AX(t) + Bu_x(0,t)\\ u_t(x,t) &= u_{xx}(x,t) + \lambda(x)X(t), \quad x \in (0,l)\\ u(0,t) &= 0\\ u(l,t) &= U(t)\\ X(0) &= X_0\\ u(x,0) &= u_0(x).\end{aligned} \tag{66}$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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